

Quantum Walks and Search Algorithms

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- 1** Notation
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Considering a two-dimensional complex vector space, we can describe it through the vectors (ket)

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (1)$$

where a unitary vector in this space is written such that

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle \quad (2)$$

considering $a_0, a_1 \in \mathbb{C}$ and

$$|a_0|^2 + |a_1|^2 = 1. \quad (3)$$

Considering a n -dimensional complex vector space, we can describe it through the vectors (ket)

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and} \quad |n-1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (4)$$

where a unitary vector in this space is written such that

$$|\psi\rangle = a_0 |0\rangle + a_1 |1\rangle + a_2 |2\rangle + \cdots + a_{n-1} |n-1\rangle = \sum_{x=0}^{n-1} a_x |x\rangle \quad (5)$$

considering $a_x \in \mathbb{C}$ and

$$\sum_{x=0}^{n-1} |a_x|^2 = 1 \quad (6)$$

We define a **bra** as the conjugate transpose of a ket. Considering a ket as column vector, we transform it into a bra such that

$$|\psi\rangle = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} \longrightarrow \langle\psi| = [a_0^* \quad a_1^* \quad a_2^* \quad \cdots \quad a_{n-1}^*]. \quad (7)$$

And the product $|\psi\rangle\langle\phi|$ is called **outer product**. For example, considering the two-dimensional space, we have

$$|0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (8)$$

Given two systems described by the vectors

$$|\psi_a\rangle = \sum_{x=0}^{n-1} a_x |x\rangle \quad \text{and} \quad |\psi_b\rangle = \sum_{y=0}^{m-1} b_y |y\rangle \quad (9)$$

we write the outer product of two vector states as

$$|\psi_a\rangle \otimes |\psi_b\rangle = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ \vdots \\ a_0 b_{n-1} \\ \vdots \\ a_{n-1} b_0 \\ a_{n-1} b_1 \\ \vdots \\ a_{n-1} b_{n-1} \end{bmatrix} \quad (10)$$

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Random Walks

- M. Dyer, A. Frieze, and R. Kannan (1991):
A random polynomial-time algorithm for approximating the volume of convex bodies.
- R. Motwani and P. Raghavan (1995):
Randomized Algorithms.
- U. Schöning (1999):
A probabilistic algorithm for k-sat and constraint satisfaction problems.
- M. Jerrum, A. Sinclair, and E. Vigoda (2004):
A polynomial-time approximation algorithm for permanent of a matrix with nonnegative entries.

Quantum Walks

- Y. Aharonov, L. Davidovich, and N. Zagury (1993):
Quantum random walks.
- E. Farhi and S. Gutmann (1998):
Quantum computation and decision trees.
- M. Szegedy (2004):
Quantum speed-up of markov chain based algorithms
- A. Patel, K. Raghunathan, and P. Rungta (2005). Quantum random walks do not need a coin toss.
- A. Childs (2009):
Universal computation by quantum walk.
- R. Portugal (2016):
The staggered quantum walk model.

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The space of our quantum walk is composed by coin \mathcal{H}_C and walker spaces \mathcal{H}_W , and we have $\mathcal{H} = \mathcal{H}_C \otimes \mathcal{H}_W$. The evolution operator consists of tossing a coin and performing a shift, and we say

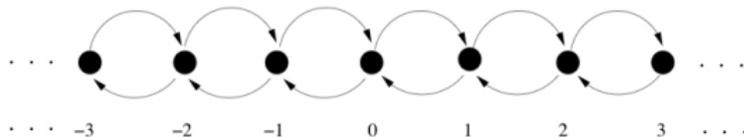
$$U = S(C \otimes I_W) \quad \longrightarrow \quad |\psi(t)\rangle = U^t |\psi(0)\rangle \quad (11)$$

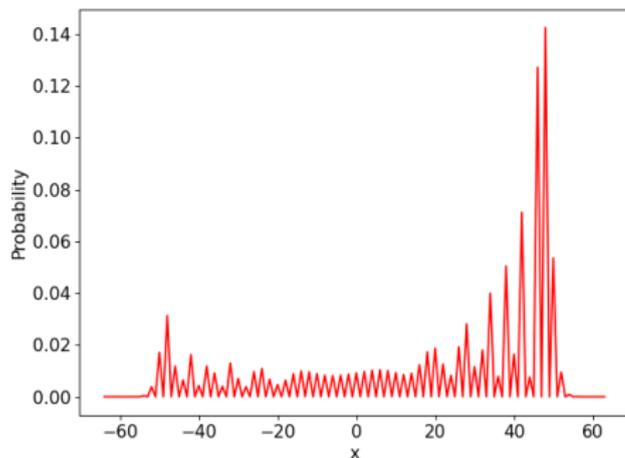
We can describe the shift operator as

$$\begin{aligned} S |0\rangle |x\rangle &= |0\rangle |x+1\rangle \\ S |1\rangle |x\rangle &= |1\rangle |x-1\rangle \end{aligned} \quad (12)$$

and S in the computational basis has the format

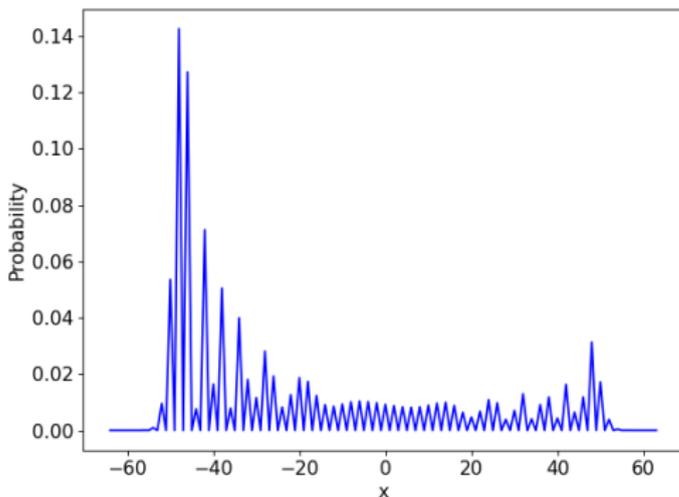
$$S = |0\rangle \langle 0| \otimes \sum_x |x+1\rangle \langle x| + |1\rangle \langle 1| \otimes \sum_x |x-1\rangle \langle x| \quad (13)$$





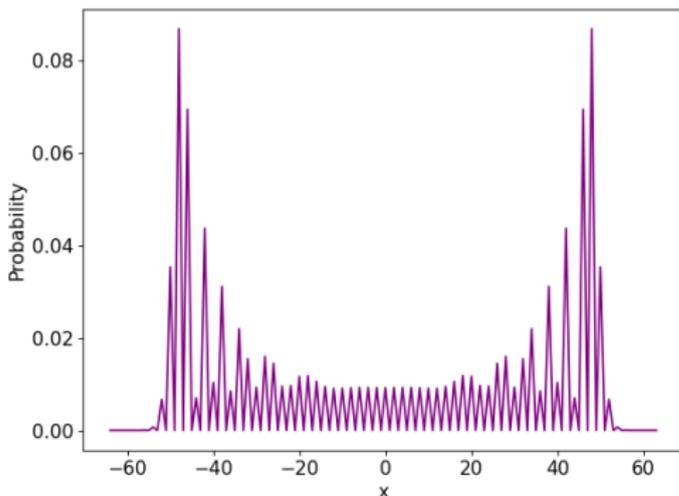
We consider $t = 70$, initial condition, and the Hadamard coin, as

$$|\psi(0)\rangle = |0\rangle |0\rangle, C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (14)$$



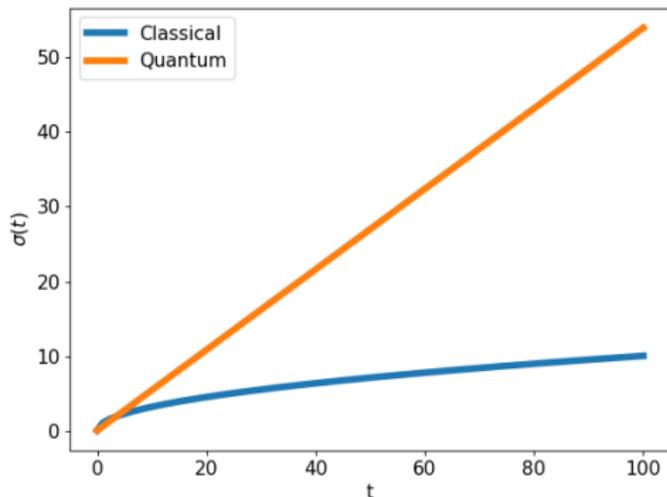
We consider $t = 70$, initial condition, and the Hadamard coin, as

$$|\psi(0)\rangle = |1\rangle |0\rangle, C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (15)$$



We consider $t = 70$, initial condition, and the Hadamard coin, as

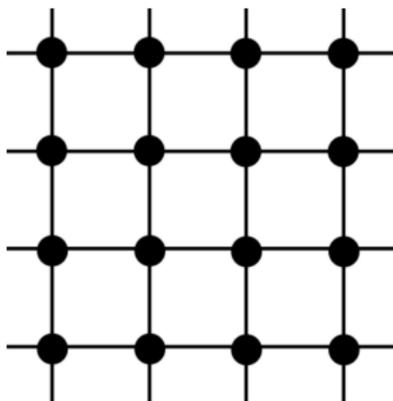
$$|\psi(0)\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}} |0\rangle, C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (16)$$



Classical: $\sigma(t) \sim \sqrt{t}$

Quantum: $\sigma(t) \sim t$

Given a square lattice, how can we describe the coined quantum walk model using this graph? How would you describe the coin and shift operators?



Given a graph G defined by its Laplacian matrix \mathcal{L} , we can define a relation between a random walk and a quantum walk by

$$\frac{\partial p(x, t)}{\partial t} = \gamma \mathcal{L} p(x, t) \quad \longrightarrow \quad i \frac{\partial |\psi(x, t)\rangle}{\partial t} = H |\psi(x, t)\rangle \quad (17)$$

Considering γ a jumping-rate (amplitude per time) and $H = -\gamma \mathcal{L}$, we have the solution

$$U = e^{i\gamma \mathcal{L} t} \quad \longrightarrow \quad |\psi(t)\rangle = U |\psi(0)\rangle \quad (18)$$

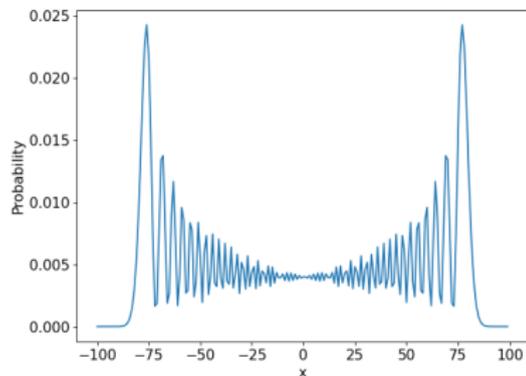
We may define the adjacency matrix for an infinite line

$$A = \sum_x |x+1\rangle \langle x| + |x\rangle \langle x+1| \quad (19)$$

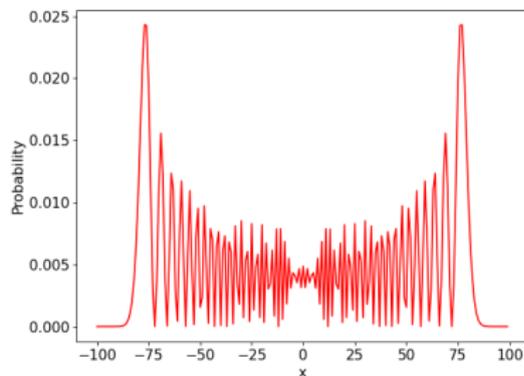
and its Laplacian matrix as

$$\mathcal{L} = A - D = \sum_x |x+1\rangle \langle x| + |x\rangle \langle x+1| - 2I \quad (20)$$

where D denotes the degree matrix of a graph.



(a) $|\psi_a\rangle$

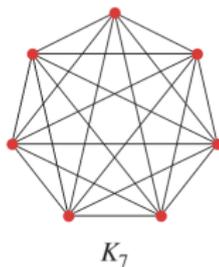
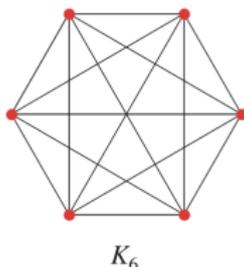
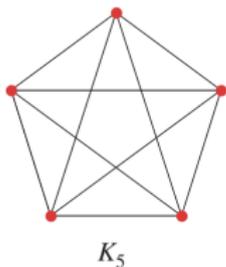


(b) $|\psi_b\rangle$

We consider here $t = 40$, $\gamma = 1$ and initial conditions

$$|\psi_a\rangle = |0\rangle \quad \text{and} \quad |\psi_b\rangle = |+\rangle \quad (21)$$

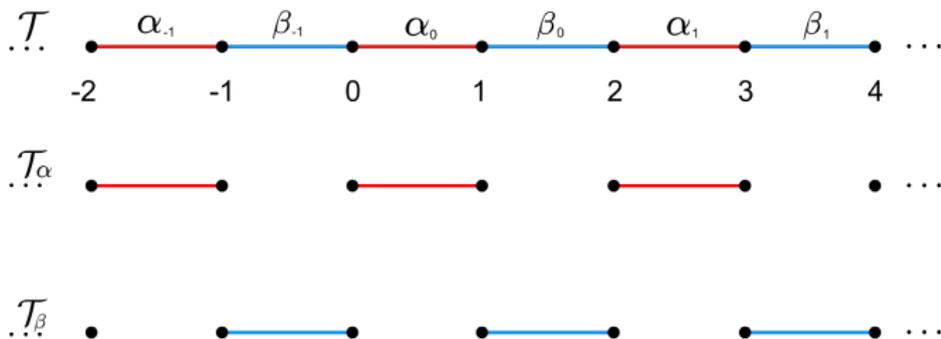
Given a clique/complete graph, how would you describe a continuous-time quantum walk considering this graph?



We construct this walk based on a graph **tessellation**, that is a set of disjoint cliques over all vertices; and a graph **covering**, that is a family of tessellations, where every edge belongs to, at least, one tessellation.

Considering an infinite line, we can tessellate this graph as

$$\begin{aligned} \mathcal{T}_\alpha &= \{\{2x, 2x + 1\} : x \in \mathbb{Z}\}, \\ \mathcal{T}_\beta &= \{\{2x + 1, 2x + 2\} : x \in \mathbb{Z}\}. \end{aligned}$$



We may define states associated to each tessellation such as

$$|\alpha_x\rangle = \frac{|2x\rangle + |2x + 1\rangle}{\sqrt{2}},$$

$$|\beta_x\rangle = \frac{|2x + 1\rangle + |2x + 2\rangle}{\sqrt{2}}.$$

and operators associated to each tessellation as

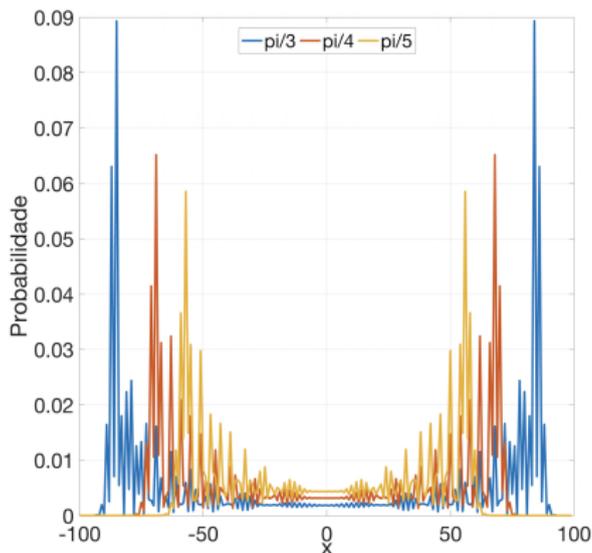
$$H_\alpha = 2 \sum_{x=-\infty}^{\infty} |\alpha_x\rangle \langle \alpha_x| - I,$$

$$H_\beta = 2 \sum_{x=-\infty}^{\infty} |\beta_x\rangle \langle \beta_x| - I.$$

Finally, we can describe the evolution operator as

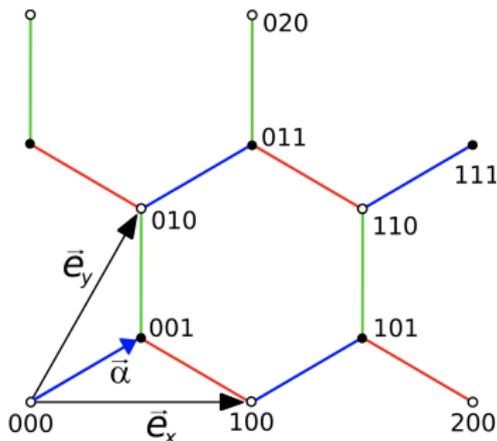
$$U = e^{i\theta_\beta H_\beta} e^{i\theta_\alpha H_\alpha}.$$

where $\theta_\alpha, \theta_\beta \in [0, \pi]$



We consider $\theta_\alpha = \theta_\beta = \theta$, after $t = 50$, and initial condition

$$|\psi(0)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad (22)$$



$$|\alpha_{x,y}\rangle = \frac{1}{\sqrt{2}} (|x, y, 1\rangle + |x + 1, y, 0\rangle)$$

$$|\beta_{x,y}\rangle = \frac{1}{\sqrt{2}} (|x, y, 1\rangle + |x, y + 1, 0\rangle)$$

$$|\gamma_{x,y}\rangle = \frac{1}{\sqrt{2}} (|x, y, 0\rangle + |x, y, 1\rangle)$$

The evolution operator for this graph will be

$$U = e^{i\theta H_\gamma} e^{i\theta H_\beta} e^{i\theta H_\alpha} \quad (23)$$

considering the operators

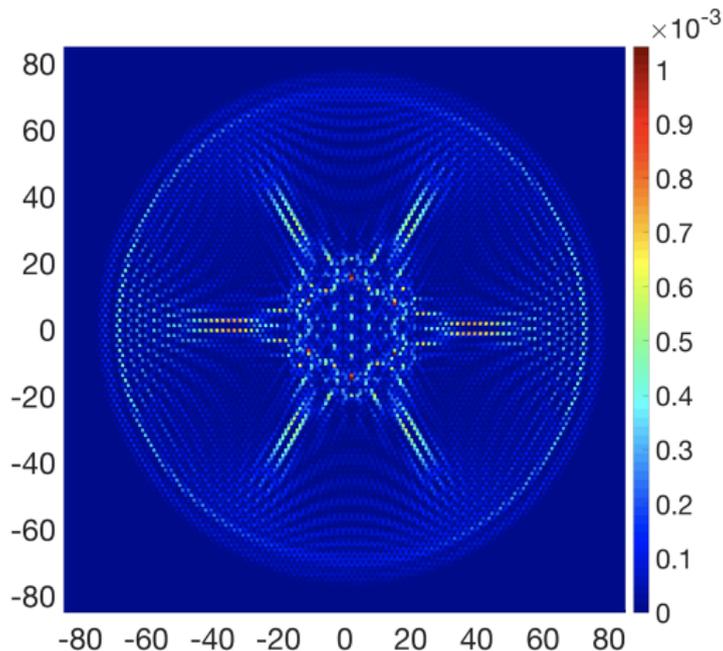
$$H_\alpha = 2 \sum_{x,y=0}^{n-1} |\alpha_{x,y}\rangle \langle \alpha_{x,y}| - I,$$

$$H_\beta = 2 \sum_{x,y=0}^{n-1} |\beta_{x,y}\rangle \langle \beta_{x,y}| - I,$$

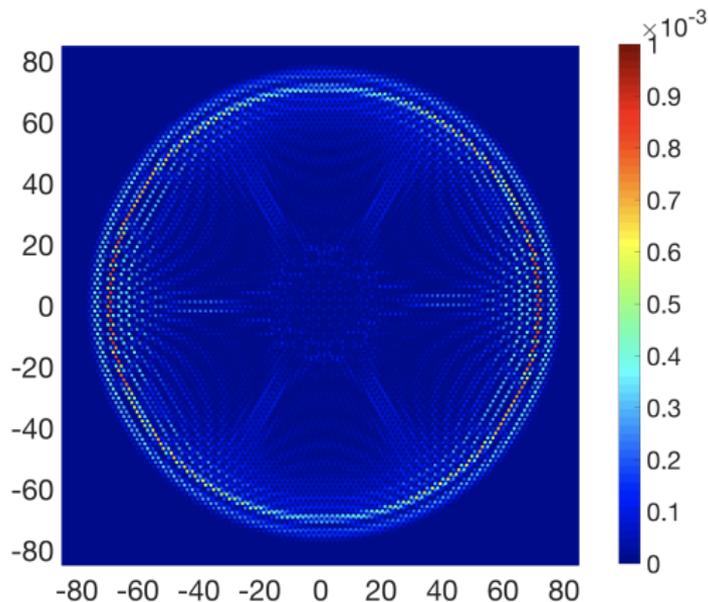
$$H_\gamma = 2 \sum_{x,y=0}^{n-1} |\gamma_{x,y}\rangle \langle \gamma_{x,y}| - I,$$

where

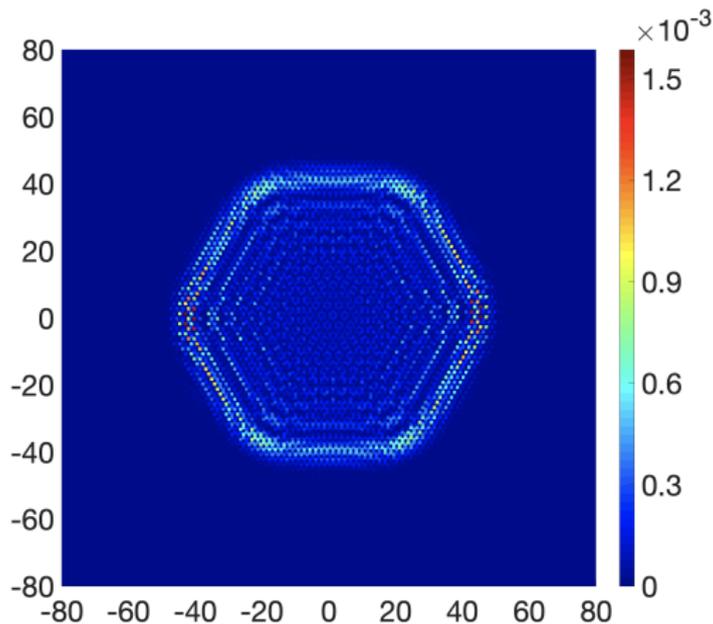
$$|\psi_t\rangle = U^t |\psi_0\rangle$$



$$|\psi_0^b\rangle = \frac{1}{\sqrt{2}}(|1, 1, 0\rangle + |1, 0, 1\rangle)$$

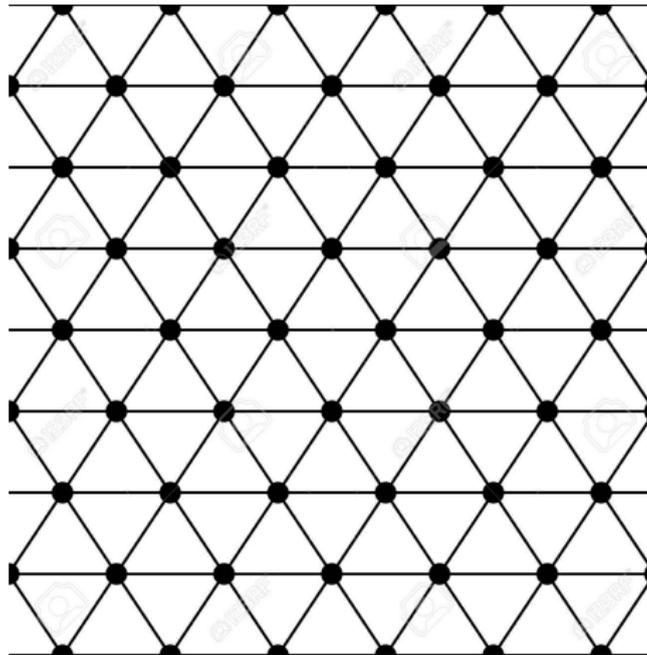


$$|\psi_0^c\rangle = \frac{1}{\sqrt{6}}(|1, 1, 0\rangle + |1, 0, 1\rangle + |1, 0, 0\rangle + |0, 0, 1\rangle + |0, 1, 0\rangle + |0, 1, 1\rangle)$$



$$|\psi_0^c\rangle = \frac{1}{\sqrt{6}}(|1, 1, 0\rangle + |1, 0, 1\rangle + |1, 0, 0\rangle + |0, 0, 1\rangle + |0, 1, 0\rangle + |0, 1, 1\rangle)$$

Given a triangle lattice, how would you tessellate this graph?



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- Triangle Finding
 - F. Magniez, M. Santha, and M. Szegedy (2003). Quantum algorithms for the triangle problem.
 - Problem: Given a graph G on n nodes, find a triangle, if there is any.
- Element Distinctness
 - A. Ambainis (2014). Quantum walk algorithm for element distinctness
 - Problem: determine whether the elements of a list are distinct.
- Matrix Product Verification
 - H. Buhrman, and R. Špalek (2005). Quantum verification of matrix products.
 - Problem: Let A, B, C be $n \times n$ matrices over any integral domain. A verification of a matrix product is deciding whether $AB = C$.
- Group Commutativity
 - F. Magniez, and A. Nayak (2005). Quantum complexity of testing group commutativity
 - Problem: Given a black-box group G with generators g_1, \dots, g_k , decide if G is abelian

Searching Algorithms: Given a list of elements, find a marked element, if there is any.

- L. Grover (1996):
A fast quantum mechanical algorithm for database search.
- M. Boyer, et al (1996):
Tight bounds on quantum searching.
- C. Zalka (1999):
Grover's quantum searching algorithm is optimal.

Searching Algorithms Based on Quantum Walks

- A. Childs, and J. Goldstone (2004). Spatial search by quantum walk.
- R. Portugal (2018). Quantum walks and search algorithms.
- J. Janmark, D. Meyer, and T. Wong (2014). Global symmetry is unnecessary for fast quantum search.
- S. Chakraborty, L. Novo, A. Ambainis, and Y. Omar. Spatial search by quantum walk is optimal for almost all graphs.

Given a graph G defined by its Laplacian matrix \mathcal{L} , we can modify the Hamiltonian

$$H = -\gamma\mathcal{L} \quad (24)$$

by introducing the oracle Hamiltonian

$$H_w = -|w\rangle\langle w| \quad (25)$$

and now on we consider the time-independent Hamiltonian,

$$H = -\gamma L + H_w = -\gamma L - |w\rangle\langle w|. \quad (26)$$

Moreover, the evolution operator will be

$$|\psi(t)\rangle = e^{itH} |s\rangle \quad (27)$$

where $|s\rangle$ denotes a uniform superposition.

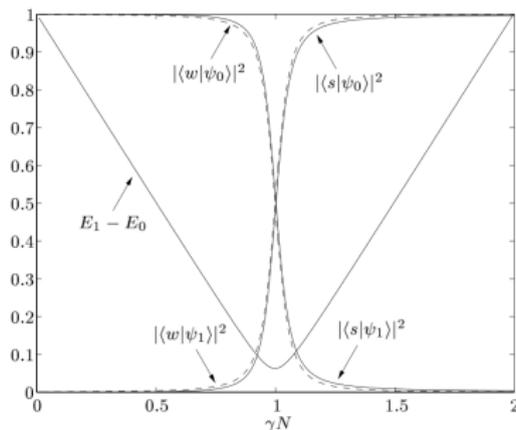
CTQW + Searching + Clique

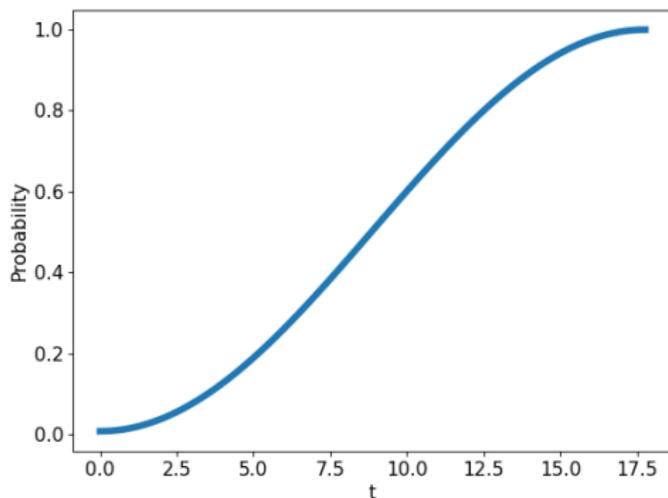


The success of the algorithm depends on the value of γ . When γ takes its critical value $1/N$, then

$$H = -|s\rangle\langle s| - |0\rangle\langle 0| \quad (28)$$

with eigenstates $|s\rangle \pm |0\rangle$ and eigenvalues $-1 \mp 1/\sqrt{N}$. So the Schrödinger evolution rotates the state from $|s\rangle$ to $|0\rangle$ in time $\pi/\Delta E = \pi\sqrt{N}$, which is optimal.





The optimal case, for one marked element over a clique graph, occurs when

$$t = \frac{\pi}{2} \sqrt{N} \quad \text{and} \quad \gamma = \frac{1}{N} \quad (29)$$

Given the evolution operator for a general coined quantum walk as

$$U = S(C \otimes I_W) \quad (30)$$

we can add an oracle unitary operator, considering the marked element w , as

$$U_w = I - 2|w\rangle\langle w| \quad (31)$$

and the searching evolution consists of

$$|\psi(t)\rangle = (UU_w)^t |s\rangle, \quad (32)$$

where $|s\rangle$ is the uniform superposition state.

Given the evolution operator for a general staggered quantum walk as

$$U = \prod_k e^{i\theta_k H_k} \quad (33)$$

we can add an oracle unitary operator, considering the marked element w , as

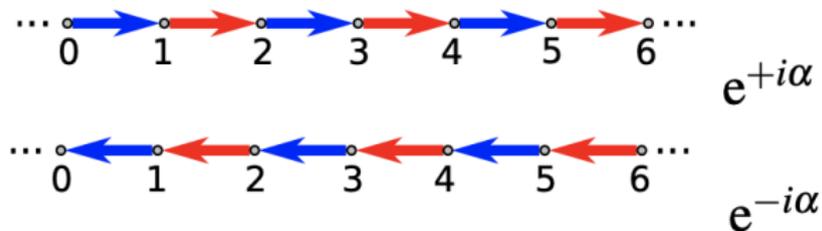
$$U_w = I - 2|w\rangle\langle w| \quad (34)$$

and the searching evolution consists of

$$|\psi(t)\rangle = (UU_w)^t |s\rangle, \quad (35)$$

where $|s\rangle$ is the uniform superposition state.

B. Chagas and R. Portugal (2020). Discrete-time quantum walks on oriented graphs.



We can modify the operators in order to have a sense of direction as

$$H_0 = \sum_x e^{-i\alpha} |2x-1\rangle \langle 2x| + e^{+i\alpha} |2x\rangle \langle 2x-1|,$$

$$H_1 = \sum_x e^{-i\alpha} |2x\rangle \langle 2x+1| + e^{+i\alpha} |2x+1\rangle \langle 2x|.$$

and the evolution will be

$$U = e^{i\theta_\beta H_1} e^{i\theta_\alpha H_0}.$$

Defining the standard deviation as $\sigma(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$, we have the moments

$$\begin{aligned} \frac{\langle x \rangle}{t} &= 2(|a|^2 - |b|^2)(1 - \cos \theta) + \frac{i \sin 2\theta (\bar{a} b e^{i\alpha} - a \bar{b} e^{-i\alpha})}{1 + |\cos \theta|} + \mathcal{O}\left(\frac{1}{t}\right) \\ \frac{\langle x^2 \rangle}{t^2} &= 4(1 - |\cos \theta|) + \mathcal{O}\left(\frac{1}{t}\right) \end{aligned}$$

considering the initial condition

$$|\psi(0)\rangle = a|0\rangle + b|b\rangle \tag{36}$$

where $|a|^2 + |b|^2 = 1$.

